ON SOME IRREGULAR FRACTIONS OF 2^n FACTORIALS

BY A. DEY AND G.M. SAHA

Institute of Agricultural Research Statistics, New Delhi

(Received in June, 1968)

1. *Introduction.* The problem of obtaining small fractions of 2^n experiments has gathered considerable attention in the recent past. Ignoring three-factor and higher order interactions Patel (1962) has given methods for constructing block designs of small fractions of 2^n factorials, where main effects are correlated with two factor interactions. Banerjee and Federer (1967) have recently proposed some plans for fractions of 2^n factorials where the main effects are uncorrelated with the two-factor interactions, but correlated among themselves. In the present paper some further irregular fractions of 2^n factorial have been discussed. Use of b.i.b. designs has been made to construct such plans. The method of analysis follows on the lines proposed by Banerjee and Federer (1967). It has been shown that under certain conditions imposed on the parameters of the b.i.b. designs the main effects as well as the two-factors interactions are mutually orthogonal among themselves ; also the main effects are uncorrelated with the two-factor interactions.

2. CONSTRUCTION

The construction of these fractional replicates is simple. Consider a b.i.b. design with parameters v, b, r, k and λ where the symbols have their usual meaning. Corresponding to such a b.i.b. design we can get another b.i.b. design which is complementary to the first one. Let $N_1=(n^1{}_{ij})$ be the incidence of matrix of the first design, *i.e.*, $n^1{}_{ij}=1$, if the jth treatment occurs in the ith block of the first b.i.b.d., otherwise, $=0(j=1,...v\,;\,i=1,...b)$

$N_2=(n^2{}_{ij})$ is a similar matrix for the complementary b.i.b. design.

Let, further, the control treatment of the v factors be denoted by (1) and its complementary (a combination with the level 1 every where) by $(\bar{1})$. Then the fraction will contain the following combinations :—

$$\begin{matrix} N_1 \\ N_2 \\ (1) \\ (\bar{1}) \end{matrix}$$

Thus, we shall get a fraction of 2^v factorial in $2b+2$ points.

3. ANALYSIS

Following Banerjee and Federer (1967), we find that without the effects containing even number of factors and ignoring the interactions with 3, 5 factors etc. the observational equations leading to the estimates of the main effects will be of the form ; (noting that a multiplying factor 1/2 is to be associated with each effect)

$$\begin{vmatrix} abd^* \\ bce^* \\ \vdots \end{vmatrix} = N_1^* \begin{vmatrix} A \\ B \\ \vdots \end{vmatrix} + \text{error part,} \qquad (3\cdot1)$$

where N_1^* is a matrix of order $(b+1)\times v$ and is obtained by replacing the unity in the matrix N_1 by a $+$ve sign and zero by a $-$ve sign together with a row with $+$ve sign everywhere. Also,

$$(abd)^* = abd - \overline{abd},$$

$$(1)^* = (1) - (\bar{1})$$

etc. Hence, the estimates of the main effects are given by (3·2), where

$$S = (N_1^{*\prime} N_1^*) :$$

$$\begin{vmatrix} A \\ B \\ C \\ \vdots \\ \vdots \\ \vdots \end{vmatrix} = S^{-1} N_1^{*1} \begin{vmatrix} abd^* \\ bce^* \\ \vdots \\ \vdots \\ \vdots \\ (1)^* \end{vmatrix} \qquad \ldots(3\cdot2)$$

The variance of any estimated main effect will be

$$\frac{1}{4(r-\lambda)}\left[\frac{(v-1)(b+1)-4(v-2)(r-\lambda)}{v(b+1)-4(v-1)(r-\lambda)}\right]2\sigma^2$$

$$=\frac{1}{2(r-\lambda)}\left[\frac{(v-1)(b+1)-4(v-2)(r-\lambda)}{v(b+1)-4(v-1)(r-\lambda)}\right]\sigma^2 \qquad \ldots(3\cdot3)$$

and the covariance between any two estimated main effects will be

$$-\frac{1}{2(r-\lambda)}\left[\frac{(b+1)-4(r-\lambda)}{v(b+1)-4(v-1)(r-\lambda)}\right]\sigma^2 \qquad \ldots(3\cdot4)$$

From (3·4), it is easy to see that the covariance between any two estimated main effects will be zero if

$$(b+1)=4(r-\lambda) \qquad \ldots(3\cdot5)$$

Hence, when (3·5) is satisfied, the estimates of main effects are uncorrelated among themselves. It is easily seen that a b.i.b. design with the following parameters satisfies (3·5).

$$v=4t-1$$
$$b=4t-1$$
$$r=2t-1$$
$$k=2t-1$$
$$\lambda=t-1$$

where t is any positive integer.

4. Two Factor Interactions

Denote $(1)+\overline{(1)}$ by $(1)_*$, $(abd)+\overline{(abd)}$ by $(abd)_*$ etc. Then the estimates of the mean response and a chosen set of $(v-1)$ two-factor interactions can be obtained on similar lines as the main effects. The N_1^* matrix in this case is of a slightly different form. Let the vector of quantities like $(1)_*$, $(abd)_*$ etc. be denoted by $M=(m_i)$. Next the vector of the mean response and interactions to be estimated is written in some order. Then the elements of the ith row of the N_1^* matrix are plus or minus according as the ith element of the M matrix has an even or odd number of letters (zero being considered as even) common with each of the elements of the vector of the mean response and two factor interactions to be estimated.

The rest of the procedure follows on the same lines as in the case of main-effects estimation. It may be mentioned here that the 2-factor interactions are orthogonal to the main effects.

5. A Case of Some Interest

Consider a b.i.b. design with following parameters :

$$v=b=s^2+s+1=n,$$

the number of factors in the 2^n experiment

$$r=k=s+1.$$
$$\lambda=1,$$

s being a prime or a prime power.

Then, if we obtain an irregular fraction through this b.i.b. design along with its complementary, it can be shown easily that our plan gives a lesser correlation (absolute) between the est. main effects than that given by Banerjee & Federer plans.

Summary

A type of fractional replicates of 2^n factorial experiments has been studied in this paper. Such small fractional replicates may find effective use in practice where for very large values of n one may be interested only in the estimation of main effects and certain two-factor interactions. For obtaining such fractions the use of balanced incomplete block designs has been suggested.

Acknowledgement

The authors are grateful to Dr. M.N. Das for his valuable criticisms during the preparation of the paper.

References

(1) Addelman, Sidney (1961) Technometrics, 3, 479-496.

(2) Banerjee, K.S. & Federer, W.T. (1967) J.R.S.S.(B), 29, 292-299:

(3) Bose, R.C. and Connor, W.S. (1960) Bull. Int. Stat. Instt. 37(3), 141-160.

(4) Patel. M.S. (1962) Ann. Math. Stat. 33, 1440-1449.